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Carlson Theorem for Harmonic Functions in $\mathbf{R}^n$*

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Boas [2] asked the following question. If u is a harmonic function of exponential type in $\mathbf{R}^3$ and $u(k, l, m) = 0$ for all integers k, l, m, under what condition of smallness of type can one conclude $u \equiv 0$? He solved the corresponding question for $\mathbf{R}^2$ and in fact the condition on the type is the same as in Carlson's theorem, namely, $\tau < \pi$. In Section 1, we deal with functions of exponential type in $\mathbf{R}^n$, and in Section 2 with those in half-space.

1

DEFINITION 1.1. A function defined in a cone $K_\alpha = \{x, x_1/r \geqslant \alpha\}$, where $-1 \leqslant \alpha < 1$ is said to be of type τ in K_α if and only if given any $\epsilon > 0$, there exists an A_ϵ such that

$$|u(x)| \leqslant A_\epsilon e^{(\tau+\epsilon)r} \qquad \text{for all } x \text{ in } K_\alpha .$$

DEFINITION 1.2. For any $x \in K_\alpha$ with $|x| = 1$,

$$h(x) = \overline{\lim_{t\to+\infty}} (\log |u(tx)|/t)$$

is called the indicator function of u in K_α.

In the case of $\mathbf{R}^2$ where u is a holomorphic function in an angle, the indicator function has nice sinusoidal properties (cf. [3], Chap. 5). But trivial examples show that indicators of harmonic functions do not have any of those nice properties.

The most that can be expected are uniqueness theorems like that of Carlson. Let us state one of the forms of Carlson's theorem. If f is a holomorphic function in the right halfplane of type $\tau < \pi$ and $f(n) = 0$ for all positive integers n, then $f \equiv 0$ on the positive x-axis and consequently $f \equiv 0$ in the right halfplane.

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In the case of harmonic functions, we can only say that they vanish on the line enclosing the zeroes and no more as again simple examples show. Thus let us start with our first theorem

THEOREM 1.3. *Let u be a harmonic function of exponential type τ in $\mathbf{R}^n$. Suppose that $\tau < \pi$ and $u(m, 0, 0,..., 0) = 0$ for all integers $m \geqslant 0$. Then $u \equiv 0$ on the x_1-axis.*

Proof. Let G denote the group of orthogonal transformation of $\mathbf{R}^n$ which fix the x_1-axis. Let us fix an $\epsilon > 0$. By hypothesis there exists an A_ϵ such that

$$| u(x)| \leqslant A_\epsilon e^{(\tau+\epsilon)|x|} \qquad \forall x \in \mathbf{R}^n.$$

Now if $T \in G$, $| u(Tx)| \leqslant A_\epsilon e^{(\tau+\epsilon)|x|}$ $\forall x \in \mathbf{R}^n$. Let μ be the normalized Haar measure on G and let $v(x) = \int_G u(Tx)\, d\mu(T)$. It is clear that v is harmonic and $| v(x)| \leqslant A_\epsilon e^{(\tau+\epsilon)|x|}$ $\forall x \in \mathbf{R}^n$. Further $u(x_1, 0,..., 0) = v(x_1, 0,..., 0)$ and consequently $v(m, 0,..., 0) = 0$ for all integers $m \geqslant 0$.

Let θ denote the angle between x and the x_1-axis. If $C_k(t)$ is the Gegenbauer Polynomial of degree k associated with $\mathbf{R}^n$, by the familiar expansion of a harmonic function in terms of its harmonics, we have

$$v(x) = \sum_{k=0}^{\infty} a_k \,| x |^k\, C_k(\cos \theta),$$

where a_k is given by $\int_{|\omega|=1} v(| x | \,\omega)\, C_k(\cos \theta)\, d\omega / | x |^k \int_{|\omega|=1} C_k^2(\cos \theta)\, d\omega$ where $d\omega$ denotes the normalized invariant measure on the unit sphere.

Hence

$$| a_k | \leqslant A_\epsilon \frac{e^{(\tau+\epsilon)|x|}}{| x |^k} \frac{\operatorname{Max} | C_k(\cos \theta)|}{\int_{|\omega|=1} C_k^2(\cos \theta)\, d\omega} \qquad \text{for all } x \in \mathbf{R}^n.$$

Taking $| x | = k/(\tau + \epsilon)$, we arrive at $| a_k | \leqslant A_\epsilon e^k k^{-k} M_k$, where

$$M_k = \frac{\operatorname{Max} | C_k(\cos \theta)|}{\int_{|\omega|=1} C_k^2(\cos \theta)\, d\omega}.$$

Presently we shall prove in Lemma 1.4 that $\operatorname{Max} | C_k(\cos \theta)| = C_k(1)$, $\lim(M_k)^{1/k} = 1$, $\lim[C_k(1)]^{1/k} = 1$. So assuming these we obtain that

$$\overline{\lim}(k/e) \,| a_k |^{1/k} \leqslant \tau + \epsilon$$

and that $\sum_0^\infty a_k C_k(1)\, z^k$ convergent power series of infinite radius and it defines an entire function of exponential type $\leqslant \tau + \epsilon$ (cf. [1], p. 130) and it vanishes on the set of nonnegative integers. Hence if $\tau < \pi$, choosing ϵ

small enough we have $\tau + \epsilon < \pi$ and Carlson's Theorem gives that $a_k C_k(1) \equiv 0$ and $C_k(1) > 0$ always, we have $v(x) \equiv 0$ and so $u(x_1, 0,..., 0) \equiv 0$.

LEMMA 1.4. *Gegenbauer Polynomial $C_k^\nu(x)$ is defined as the coefficient of h^k in $(1 - 2xh + h^2)^{-\nu}$. We assert that* $\operatorname{Max}_{-1\leqslant x\leqslant 1} | C_k^\nu(x)| = C_k^\nu(1)$ *and* $\lim_{k\to\infty}[C_k^\nu(1)]^{1/k} = 1$. *Further* $\lim_{k\to\infty}[\int_{|\omega|=1} C_k^2(\cos\theta)\, d\omega]^{1/k} = 1$.

Proof. Write $x = \cos\theta$ and $1 - 2xh + h^2 = (1 - e^{i\theta}h)(1 - e^{-i\theta}h)$. For small h we use the binomial expansion and obtain that

$$C_k^\nu(1) = \sum_{r+s=k} e^{i(r\theta - s\theta)} \frac{\nu(\nu+1)\cdots(\nu+r-1)\,\nu(\nu+1)\cdots(\nu+s-1)}{r!s!}.$$

Clearly

$$| C_k^\nu(x)| \leqslant C_k^\nu(1) = \sum_{r+s=k} \frac{\nu(\nu+1)\cdots(\nu+r-1)\,\nu(\nu+1)\cdots(\nu+s-1)}{r!s!}.$$

On the other hand $C_k^\nu(1)$ is the coefficient of h^k in $(1-h)^{-2\nu}$, i.e., $2\nu(2\nu+1)\cdots(2\nu+k-1)/k!$. Now by Stirling's formula

$$\lim_{k\to\infty} [\max_{-1\leqslant x\leqslant 1} | C_k^\nu(x)|]^{1/k} = \lim_{k\to\infty} [| C_k^\nu(1)|]^{1/k} = 1.$$

Further the Jacobi Polynomial $P_k^{\alpha,\alpha}(x)$ (cf. [4], Chap. IV) is a constant multiple of $C_k^\nu(x)$ where $\alpha = \nu - \frac{1}{2}$. Therefore using again Stirling's formula and expressions for $P_k^{\alpha,\alpha}(1)$ and $\int_{|\omega|=1} P_k^2(\cos\theta)\, d\omega$ (as given in [4], Chap. IV), we obtain that

$$\lim_{k\to\infty} \left[\frac{P_k^{\alpha,\alpha}(1)}{\int_{|\omega|=1} P_k^2(\cos\theta)\, d\omega}\right]^{1/k} = 1$$

and consequently

$$\lim_{k\to\infty} \left[\int_{|\omega|=1} (C_k^\nu(\cos\theta))^2\, d\omega\right]^{1/k} = 1.$$

COROLLARY 1.5. *Type of the symmetrization v of u (as in Theorem 4.3) is* $\overline{\lim}_{k\to\infty}(k/e) | a_k |^{1/k}$ *which in turn is equal to the type of holomorphic extension of the real analytic function u restricted to the x_1-axis. (From here on we shall refer to this as the complexification of u on x_1-axis.)*

Proof. We have already proved that $\overline{\lim}_{k\to\infty}(k/e) | a_k |^{1/k} \leqslant$ type of v. For the converse we observe that

$$| v(x)| \leqslant \sum_0^\infty | a_k | \operatorname{Max} | C_k(\cos\theta)| \, | x |^k = \sum_{k=0}^\infty b_k | x |^k$$

and $\overline{\lim}_{k\to\infty}(k/e) \mid b_k \mid^{1/k} = \overline{\lim}_{k\to\infty} \mid a_k \mid^{1/k}$. Now corollary follows from pp. 130–131 of [1].

COROLLARY 1.6. *u is harmonic function in $\mathbf{R}^n$ of type $\tau < \pi$, and vanishes for u lattice points $(x_1, ..., x_n)$ where $x_1, x_2, ..., x_n$ are nonnegative integers, then $u \equiv 0$.*

Proof. Translation does not change the type and consequently we obtain that u vanishes on all the lines passing through lattice points and parallel to the axes. This would in turn give us that u vanishes all planes passing through lattice points and normal to the axes. This implies that $u \equiv 0$.

THEOREM 1.7. *Let u be any analytic function in $\mathbf{R}^n$ such that $\Delta u \equiv \lambda^2 u$, where Δ is the Laplace operator. Let τ be the type of u. If $\tau < |\lambda|$, then $u \equiv 0$. In other words $u \equiv 0$ or type of $u \geqslant |\lambda|$.*

Proof. Let T be any orthogonal transformation of $\mathbf{R}^n$. Then $U(Tx)$ also satisfies $\Delta u \equiv \lambda^2 u$. Hence if $\tilde{G}$ is the group of orthogonal transformations of $\mathbf{R}^n$ and $\tilde{\mu}$ its normalized Haar measure, then $\tilde{v}(x) = \int_{\tilde{G}} u(Tx)\, d\mu(T)$ also satisfies $\Delta u = \lambda^2 u$ and depends only on $|x|$. Thus $\tilde{v}(x) = \tilde{v}(r)$ satisfies the differential equation

$$\frac{du^2}{dr^2} + \frac{n-1}{r}\frac{du}{dr} - \lambda^2 u = 0.$$

Consequently $\tilde{v}(r) = \sum_0^\infty a_{2k} r^{2k}$ when $(2k+2)(2k+n)\, a_{2k+2} = \lambda^2 a_{2k}$ and so

$$a_{2k} = \frac{\lambda^{2k} \cdot a_0}{2k(2k+n(2k-2)(2k+n-2) \cdots (2 \cdot n)}.$$

Therefore $\lim_{k\to\infty}(2k/e)(a_{2k})^{1/2k} = \lambda$ by Stirling's Formula. Thus type of $\tilde{v}(r) = \lambda$ if $a_0 \neq 0$ and on the other hand type of $\tilde{v} \leqslant$ type of $u = \tau$ which implies that $\tau < |\lambda|$ implies $v(r) \equiv 0$, i.e., $u(0) = \tilde{v}(0) = 0$. Again type being independent of translation, we arrive at $u \equiv 0$. Q.E.D.

COROLLARY 1.8. *If u is a harmonic function of type $\tau < \pi$ and vanishes on all lattice points in the planes $x_1 = 0$ and $x_1 = 1$, then $u \equiv 0$.*

Proof. By the same argument as in Corollary 1.6, we find that $u \equiv 0$ on the planes $x_1 = 0$ and $x_1 = 1$. Let X_1 denote $(x_2, x_3, ..., x_n)$ and $x = (x_1, X_1)$. Now it is clear that $u(x) \equiv \sum_{k=1}^\infty A_k(X_1) \sin k\pi x_1$, where $A_k(X_1)$ are of type $\leqslant \tau$ and they satisfy the differential equation $\Delta_{X_1} A_k = k^2\pi^2 A_k$, where Δ_{X_1} is the Laplace operator in the variables $x_2, x_3, ..., x_n$. By Theorem 1.7, we get that $A_k \equiv 0$ for $k \geqslant 1$ which proves the corollary.

Remark. It is now obvious that the corollary is the best possible. There exist functions of type $\tau = \pi$ which vanish at all lattice points in $\mathbf{R}^n$ but not identically zero say $A_1(X_1) \sin \pi x_1$.

2

Carlson's Theorem is stated for the half-plane and in Section 1 we proved analogous theorems in the whole of $\mathbf{R}^n$. Now we shall deal with the case of half-space. We can only expect theorems like 1.3 and no better.

We start with a few notations. K_α denotes the case $\{x; x_1/r \geqslant \alpha\}$ where $-1 \leqslant \alpha < 1$. Given any harmonic function in K_α, we shall define two operators Su and Cu. Su is a harmonic function in K_α defined by $Su(x) = \int_G u(Tx)\, d\mu(T)$, where G is the group of orthogonal transformations of $\mathbf{R}^n$ which leave x_1-axis fixed and μ is the normalized Haar measure on G. $Cu(z)$ is a holomorphic function defined in the angle $\Gamma_\alpha = \{re^{i\theta}; |\theta| \leqslant \cos^{-1}\alpha\}$ and it is obtained by analytic continuation of $u(x, 0, 0,..., 0)$ on the positive real axis of the complex plane. We shall soon see that it can be analytically continued to the whole of Γ_α.

Remark 2.1. If u is of type τ in K_α, so is Su in K_α.

Remark 2.2. $Su = 0$ on the axis of K_α implies $Su \equiv 0$, i.e., the mapping $Su \to Cu$ is well-defined. This can be observed by expanding Su in terms of Gegenbauer functions in K_α.

The same method would give us that Su restricted to the real axis can be continued analytically in any disc with center on the real axis and entirely contained in Γ_α and also the fact that $Su \to Cu$ is an invertible mapping.

Remark 2.3. If u is harmonic and of exponential type τ in K_α, then any of its derivatives is also of exponential type τ in $K_\alpha + \epsilon$, where $K_\alpha + \epsilon$ is the cone K_α pushed in the direction of x_1-axis by ϵ.

THEOREM 2.4. *If u is harmonic in the half-space K_0 and of exponential type $\tau < \pi$ and further $u(m, 0,..., 0) = 0$ for all integers m in the axis of K_0, then $u \equiv 0$ on the axis.*

Proof. For simplicity let us do it for $\mathbf{R}^3$. Let $D_{\epsilon,R}$ denote the domain $\{x; x_1 > \epsilon \text{ and } |x - \epsilon| < R\}$, where ϵ denotes the point $(\epsilon, 0,..., 0)$. Applying Green's formula we obtain for t on the axis

$$u(t) = \int_{\partial D_{\epsilon,R}} \frac{1}{|t-x|}\frac{\partial u}{\partial n} - \frac{\partial}{\partial n}\frac{1}{|t-x|} \cdot u \, d\sigma(x),$$

where $\epsilon < t < R + \epsilon$ and $\partial/\partial n$ denotes unit outward normal and $d\sigma(x)$ is the area measure on $\partial D_{\epsilon,R}$. If θ is the angle between the axis and x, then $| t - x |^2 = t^2 - 2 | x | t \cos \theta + | x |^2 = (t - | x | e^{i\theta})(t - | x | e^{-i\theta})$. We can replace t by the complex variable z and then

$$| z - x | = (z^2 - 2 | x | z \cos \theta + | x |^2)^{-1/2}$$

is holomorphic in the region defined by $\epsilon < \operatorname{Re} z < R + \epsilon$ and $| z - \epsilon | < R$ for any $x \in \partial D_{\epsilon,R}$. Thus we obtain

$$Cu(z) = \int_{\partial D_{\epsilon,R}} \frac{1}{| z - x |} \frac{\partial u}{\partial n} - \frac{\partial}{\partial n} \frac{1}{| z - x |} \cdot u \, d\sigma(x).$$

We fix ϵ and let $R \to \infty$. We get $Cu(z)$ holomorphic in $\Gamma_0 + \epsilon$ and of type $\tau < \pi$ in $\Gamma_0 + \epsilon$. Further $Cu(m) = u(m, 0, 0,..., 0) = 0$ for all positive integers m. Hence by Carlson's theorem $Cu \equiv 0$ which implies $u \equiv 0$ on the axis of K_0. Q.E.D.

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